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Singular vortex in magnetohydrodynamics

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Abstract

We obtain an exact solution of the ideal magnetohydrodynamics equations in (3+1) dimensions. The construction of the solution is based on the invariants of the group $O(3)$ of rotations. It is a deep generalization of the known solution, which describes a radial flow with spherical waves. It is shown that in the irreducible solution, the velocity and magnetic field vectors of the particle are coplanar to the radius vector of the particle. The solution is represented as an involutive invariant subsystem of PDEs with two independent variables and finite relation, which contains a functional arbitrariness.

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1. Introduction

In this paper, we construct an exact solution to the ideal magnetohydrodynamics equations. The solution is obtained on the concept of the 'singular vortex', which was recently developed on the basis of the symmetry approach. From the group-theoretical point of view, the singular vortex is a partially invariant solution with defect 1, constructed on the group of rotations in three-dimensional space. The main difficulty in the procedure of construction of partially invariant solutions is a completion to involution of the overdetermined system for non-invariant functions. It is a branching process, which sometimes becomes quite complicated. In this paper the investigation of the singular vortex is carried out completely. The solution is described in terms of a finite integral, which contains a functional arbitrariness, and in terms of the involutive reduced system of PDEs with two independent variables. The paper describes a purely mathematical construction of the solution. Here we do not deal with the physical features and interpretation of the plasma motion, governed by the singular vortex. The latter will become an object of investigation for other papers, which we hope to present in the near future.

The first section of the paper represents a brief description of the singular vortex concept. The subsequent sections contain the analysis of the singular vortex in the ideal magnetohydrodynamics equations.

2. The singular vortex: general concept

In the space $\mathbb{R}^3(x, y, z) \times \mathbb{R}^3(u, v, w)$, a group $O(3)$ of simultaneous rotations of the subspaces $\mathbb{R}^3(x, y, z)$ and $\mathbb{R}^3(u, v, w)$ is given. The corresponding Lie algebra is generated by the following operators:

$$\begin{aligned} X_1 &= z\partial_y - y\partial_z + w\partial_v - v\partial_w, \\ X_2 &= x\partial_z - z\partial_x + u\partial_w - w\partial_u, \\ X_3 &= y\partial_x - x\partial_y + v\partial_u - u\partial_v. \end{aligned} \tag{2.1}$$

Here and below $\partial_x \equiv \partial/\partial x$. In order to check the necessary conditions of existence of $O(3)$-invariant solution, the matrix of coefficients of operators (2.1) is to be written as

$$M(\xi\,|\,\eta) = \left(\begin{array}{ccc|ccc} 0 & z & -y & 0 & w & -v \\ -z & 0 & x & -w & 0 & u \\ y & -x & 0 & v & -u & 0 \end{array}\right).$$

One can easily verify that

$$\operatorname{rank} M(\xi) < \operatorname{rank} M(\xi\,|\,\eta). \tag{2.2}$$

Here $M(\xi)$ is a matrix of the first three columns of $M(\xi\,|\,\eta)$. Relation (2.2) proves a well-known fact [2, 3] that a non-singular $O(3)$-invariant solution of any system of equations for the sought functions $\boldsymbol{u} = (u, v, w)$ and independent variables $\boldsymbol{x} = (x, y, z)$ does not exist. However, one can use an ansatz

$$\boldsymbol{u} = f(|\boldsymbol{x}|)\boldsymbol{x}, \tag{2.3}$$

which corresponds to a singular $O(3)$-invariant solution. Relations (2.3) define a singular manifold of the group $O(3)$ since $\operatorname{rank} M(\xi) = \operatorname{rank} M(\xi\,|\,\eta) = 2$ in the points of the manifold (2.3). It is also the invariant manifold as long as $X_j(\boldsymbol{u} - f(|\boldsymbol{x}|)\boldsymbol{x}) = 0$ whenever (2.3) holds ($j = 1, 2, 3$). Solutions of the type (2.3) are usually called the rotationally invariant ones.

Group $O(3)$ gives rise to another type of solution, namely, a partially invariant one. Let us observe a spherical coordinate system

$$x = r\sin\theta\cos\varphi, \qquad y = r\sin\theta\sin\varphi, \qquad z = r\cos\theta. \tag{2.4}$$

The decomposition of a vector field $\boldsymbol{u}$ on the basis of a spherical coordinate system gives

$$\begin{aligned} u_r &= u\sin\theta\cos\varphi + v\sin\theta\sin\varphi + w\cos\theta, \\ u_\theta &= u\cos\theta\cos\varphi + v\cos\theta\sin\varphi - w\sin\theta, \\ u_\varphi &= -u\sin\varphi + v\cos\varphi. \end{aligned} \tag{2.5}$$

With these notations, invariants of the $O(3)$ can be written as follows:

$$r = |\boldsymbol{x}|, \qquad u_r, \qquad u_\theta^2 + u_\varphi^2.$$

The representation of invariant part of solution is distinguished by the following conditions:

$$u_r = U(r), \qquad u_\theta^2 + u_\varphi^2 = M^2(r). \tag{2.6}$$

Only two of the three functions, which determine a vector field $\boldsymbol{u}$, are specified by the equalities (2.6). The third value is assumed to be an arbitrary function of (r, θ, φ), namely

$$\begin{aligned} u_\theta &= M\cos\omega, \qquad u_\varphi = M\sin\omega, \\ \omega &= \omega(r, \theta, \varphi). \end{aligned} \tag{2.7}$$

For any system of equations, which admits the Lie group $O(3)$, relations (2.6), (2.7) define a representation of the $O(3)$ partially invariant solution. Functions U and M will be called hereafter invariant ones since they depend only on the invariant variable r. In contrast, function ω, which depends on all the independent variables (r, θ, φ), will be called non-invariant.

Note that any other sought functions, which are not transformed under $O(3)$ action must be assumed as invariant, i.e. dependent only on the invariant variable r. On the other hand, if equations involve some additional independent variables, for example, time t, the dependence on these variables must be added to both invariant and non-invariant functions.

The substitution of the obtained representation of the solution into the investigated system of equations usually gives equations of two types: invariant ones, which involve only invariant functions and variables, and non-invariant ones, which involve the non-invariant function ω. The latter equations should be observed as an overdetermined system for the non-invariant function ω. Its compatibility conditions enlarge the invariant subsystem. Solution of the invariant subsystem and consequent determination of the non-invariant function give the solution of the initial equations.

First, the solution of the type (2.6), (2.7) was investigated by Ovsyannikov [1] for Euler equations for an ideal compressible and incompressible fluid. In his work, the overdetermined system for function ω was completed to involution. Its general implicit solution, which involves an arbitrary function of two arguments, was also given. All the invariant functions were determined from the well-defined system of PDEs with two independent variables. The main features of the fluid flow, governed by the obtained solution, were pointed out. Namely, it was shown that trajectories of particles are flat curves in three-dimensional space. The position and orientation of the plane, which contains the trajectory, depend on the particle's initial location. Another noted feature is that the continuous solution can be determined not in the whole space, but in some moving or stationary channels.

The title of Ovsyannikov's article 'singular vortex' is related to the solution with a special choice of the non-invariant function, which guarantees the continuous initial data. Afterwards, the name 'singular vortex' was assigned to all solutions, which are partially invariant with respect to the group $O(3)$.

Independent investigation of the $O(3)$ partially invariant solution for ideal incompressible fluid was performed by Popovych [4]. The article includes the investigation of the overdetermined system for non-invariant function and also the investigation of symmetry reductions of the invariant subsystem. Further analysis of the singular vortex for an ideal compressible fluid can be found in [5, 6].

The general concept of singular vortex was proposed by L V Ovsyannikov at his lecture at the conference 'New mathematical models at mechanics: construction and investigation', which was held on May 10–14, 2004 in Novosibirsk, Russia. Ovsyannikov has also shown examples of an acoustic singular vortex and an irrotational singular vortex. According to the suggestion by the corresponding member of the Russian Academy of Science, S I Pohozhaev, the singular vortex is now sometimes called the 'Ovsyannikov vortex'.

In this work, we investigate a singular vortex for the mathematical model of ideal compressible magnetohydrodynamics. The analysis is complicated by simultaneous presence of two vector fields: velocity and magnetic. The system for the non-invariant function ω is strongly overdetermined but it is possible to find a condition under which the system is in involution and has a functional arbitrariness of the solution. The latter condition is that for any particle of fluid its radius vector, velocity and magnetic field vectors are coplanar. In this case the non-invariant function is determined from the implicit finite (not differential) equation, which involves one arbitrary function of one argument.

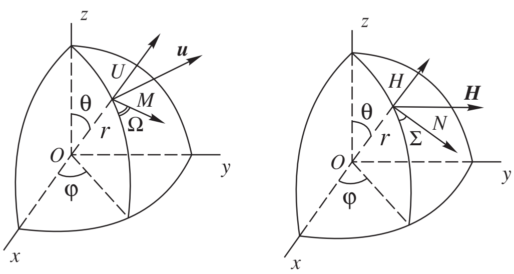

Figure 1. The decomposition of the velocity and magnetic field vectors.

A review of other symmetry reductions of MHD equations can be found in the handbook [8]. The most recent papers related to the construction of MHD exact solutions are [9, 10]. In particular, the latter contains the $O(3)$ singular invariant solution of the form (2.3).

3. Preliminary information

The equations for an ideal fluid with infinite conductivity are the following:

$$\begin{aligned}
&D\rho + \rho \operatorname{div} \boldsymbol{u} = 0,\\
&D\boldsymbol{u} + \rho^{-1}\nabla p + \rho^{-1}\boldsymbol{H} \times \operatorname{rot} \boldsymbol{H} = 0,\\
&Dp + A(p, \rho) \operatorname{div} \boldsymbol{u} = 0,\\
&D\boldsymbol{H} + \boldsymbol{H} \operatorname{div} \boldsymbol{u} - (\boldsymbol{H} \cdot \nabla)\boldsymbol{u} = 0,\\
&\operatorname{div} \boldsymbol{H} = 0, \qquad D = \partial_t + \boldsymbol{u} \cdot \nabla.
\end{aligned} \tag{3.1}$$

Here $\boldsymbol{u} = (u, v, w)$ is the velocity vector, p, ρ are pressure and density, $\boldsymbol{H} = (H^1, H^2, H^3)$ is the magnetic field. All functions depend on time t and coordinates (x, y, z). Function $A(p, \rho)$ depends on the state equation of the fluid. Note that system (3.1) is overdetermined; it contains nine equations for eight sought functions. However, the system (3.1) is in involution since the last equation can be observed as a restriction for the initial data. According to the induction equation if the last equation is satisfied at some moment of time, then it will also be valid for all times of existence of the solution.

The admitted group for the system (3.1) for the case of polytropic state equation $A(p, \rho) = \gamma p$ (γ is the adiabatic exponent) is known [7, 8]. It is a 13-dimensional extension of the Euclidean group via time translation and dilatation.

The admitted group includes a simple subgroup $O(3)$ of simultaneous rotations in the spaces $\mathbb{R}^3(\boldsymbol{x})$, $\mathbb{R}^3(\boldsymbol{u})$ and $\mathbb{R}^3(\boldsymbol{H})$. Construction of the singular vortex for equations (3.1) demands calculation of invariants of $O(3)$ in the space of functions and variables.

4. The representation of the solution

For convenience, we observe the spherical coordinate system (2.4). Vectors $\boldsymbol{u}$ and $\boldsymbol{H}$ are decomposed by spherical frame according to (2.5). The following individual notations of components of velocity and magnetic field vectors are introduced (see figure 1):

$$\begin{aligned}
&v_r = U, \qquad v_\theta = M\cos\Omega, \qquad v_\varphi = M\sin\Omega;\\
&H_r = H, \qquad H_\theta = N\cos\Sigma, \qquad H_\varphi = N\sin\Sigma.
\end{aligned} \tag{4.1}$$

Here U and H are radial components of $\boldsymbol{u}$ and $\boldsymbol{H}$. Functions M and N denote the absolute values of its components tangential to spheres r = const. Functions Ω and Σ are the angles between the tangential components of $\boldsymbol{u}$ and $\boldsymbol{H}$ and the meridional direction.

In these notations the invariants of the group $O(3)$ can be chosen as follows:

$$t, r, U, M, H, N, \Omega - \Sigma, p, \rho. \tag{4.2}$$

According to the described algorithm, the representation of the partially invariant solution is constructed in the form

$$\begin{aligned} &U = U(t,r), && M = M(t,r), && H = H(t,r), && N = N(t,r), \\ &\Sigma = \sigma(t,r) + \omega(t,r,\theta,\varphi), && \Omega = \omega(t,r,\theta,\varphi), && p = p(t,r), && \rho = \rho(t,r). \end{aligned} \tag{4.3}$$

Substitution of the representation (4.3) into system (3.1) gives equations of the singular vortex.

5. The reduced equations

First of all let us observe a transformation of continuity equation (first equation of (3.1)) under the substitution (4.3). It is convenient to introduce a new invariant function $h(t,r)$, defined by the equality

$$D_0\rho + \rho(r^{-2}(r^2U)_r - r^{-1}hM) = 0, \qquad D_0 = \partial_t + U\partial_r. \tag{5.1}$$

Here and below the lower indices t, r, θ and φ denote the corresponding partial derivatives. From the continuity equation there follows the equation for the non-invariant function ω:

$$\sin\theta \sin\omega\omega_\theta - \cos\omega\omega_\varphi - h\sin\theta - \cos\theta\cos\omega = 0. \tag{5.2}$$

The function h plays a crucial role in description of a fluid motion, governed by the obtained solution.

There are another two equations, which involve only invariant functions. These are

$$\begin{aligned} &D_0U + \rho^{-1}(p_r + N(N)_r + r^{-1}N^2) - r^{-1}M^2 = 0, \\ &D_0p + A(p,\rho)(r^{-2}(r^2U)_r - r^{-1}hM) = 0, \qquad D_0 = \partial_t + U\partial_r. \end{aligned} \tag{5.3}$$

The first equation in (5.3) follows from the radial momentum equation, and the second equation in (5.3) is the energy equation. Another six equations of the system (3.1) with (4.3) substituted sufficiently depend on function ω and its derivatives.

$$\begin{aligned} &HN\sin\sigma\omega_r + r^{-1}N^2\sin\sigma\cos(\sigma+\omega)\omega_\theta + r^{-1}N^2\sin\sigma\sin^{-1}\theta\sin(\sigma+\omega)\omega_\varphi \\ &\qquad + \rho D_0M + HN\sin\sigma\sigma_r - H\cos\sigma(N)_r + r^{-1}MU\rho \\ &\qquad + N^2\cot\theta\sin\sigma\sin(\sigma+\omega) - r^{-1}HN\cos\sigma = 0. \end{aligned} \tag{5.4}$$

$$\begin{aligned} &M\omega_t + (MU - \rho^{-1}HN\cos\sigma)\omega_r + r^{-1}(M^2\cos\omega - \rho^{-1}N^2\cos\sigma\cos(\sigma+\omega))\omega_\theta \\ &\qquad - (r\rho\sin\theta)^{-1}(N^2\cos\sigma\sin(\sigma+\omega) - M^2\rho\sin\omega)\omega_\varphi \\ &\qquad - \rho^{-1}HN\sigma_r - H\sin\sigma(N + r(N)_r) \\ &\qquad - (r\rho)^{-1}N^2\cot\theta\cos\sigma\sin(\sigma+\omega) + r^{-1}M^2\sin\omega = 0. \end{aligned} \tag{5.5}$$

$$\begin{aligned} &\sin\theta(-HM\sin\omega + NU\sin(\sigma+\omega))\omega_\theta + (HM\cos\omega - NU\cos(\sigma+\omega))\omega_\varphi \\ &\qquad + r\sin\theta H_t - NU\cos\theta\cos(\sigma+\omega) + HM\cos\theta\cos\omega = 0. \end{aligned} \tag{5.6}$$

$$HM\sin\sigma\omega_r - D_0N + M\cos\sigma H_r - r^{-1}N(rU)_r + r^{-1}H\cos\sigma(rM)_r = 0. \tag{5.7}$$

$$N\omega_t + (NU - HM\cos\sigma)\omega_r + ND_0\sigma + r^{-1}\sin\sigma(rHM)_r = 0. \tag{5.8}$$

$$-N\sin\theta\sin(\sigma+\omega)\omega_\theta + N\cos(\sigma+\omega)\omega_\varphi + \sin\theta r^{-1}(r^2H)_r + N\cos\theta\cos(\sigma+\omega) = 0. \tag{5.9}$$

Equations (5.4), (5.5) are the momentum equations in the direction tangential to sphere $r = \text{const}$. The next equations (5.6)–(5.8) follow from the induction equations in projections to the radial and tangential directions. Finally, equation (5.9) is Gauss' law $\operatorname{div}\mathbf{H} = 0$.

Equations (5.2), (5.4)–(5.9) form an overdetermined quasilinear system for the non-invariant function ω. Its compatibility conditions enlarge the subsystem (5.1), (5.3) for the invariant functions.

6. The irreducibility conditions

Now we restrict our investigations to the case of irreducible partially invariant solutions [2]. This means that we demand the solutions (4.3) not to be invariant of rank 2 with respect to any subgroup of the group, admissible by equations (3.1). According to the sufficient condition of reducibility of partially invariant solutions [2], this leads to the following. It would be impossible to express all the first-order derivatives of the function ω from the overdetermined systems (5.2), (5.4)–(5.9). In other words, this condition means that the function ω is determined by equations (5.2), (5.4)–(5.9) with a functional arbitrariness. To check the irreducibility condition one has to write a matrix of coefficients of ω's derivatives in systems (5.2), (5.4)–(5.9) and, then, to remove all minors of rank 4 of this matrix. The direct computations show that only four cases are possible when the matrix of coefficients has rank 3 or less, namely

$$1.\quad N=0, \qquad 2.\quad H=0, \qquad 3.\quad M=0, \qquad 4.\quad \sigma=0. \tag{6.1}$$

Note that case 4 of (6.1) corresponds to $\Omega = \Sigma$. It means that the radius vector, velocity and magnetic field vectors at any point are coplanar. Cases 1 and 3 can be considered as special degenerate subcases of 4, when the magnetic field vector or velocity vector is collinear to the radius vector of the point. In any case, we will consider all four possibilities (6.1) separately in order to simplify the analysis of the equations. In these investigations we omit possibilities which lead to known solutions: pure gas dynamics $\boldsymbol{H} = 0$ or radial solution of the type (2.3) $M = N = 0$. Also, for physical reasons we consider density ρ and pressure p to be positive functions: $\rho > 0$, $p > 0$.

7. Radial magnetic field

Let $N \equiv 0, M \neq 0$. The number of invariant equations (5.1), (5.3) is increased by the following equations. From (5.4) it follows that

$$D_0M + r^{-1}UM = 0. \tag{7.1}$$

Equation (5.6), taking into account (5.2), gives

$$H_t - r^{-1}hHM = 0. \tag{7.2}$$

Multiplying (5.7) by $\cos\sigma$, (5.8) by $\sin\sigma$ and summing lead to

$$(rHM)_r = 0. \tag{7.3}$$

Finally, equation (5.9) simplifies to

$$(r^2H)_r = 0. \tag{7.4}$$

The overdetermined system for ω in this case consists of equation (5.2) and two equations, which follow from (5.5) and from linear combination of (5.7) and (5.8). The latter two equations for ω are

$$r\sin\theta M^{-1}\omega_t + r\sin\theta U M^{-1}\omega_r + \sin\theta\cos\omega\omega_\theta + \sin\omega\omega_\varphi + \cos\theta\sin\omega = 0, \tag{7.5}$$

$$\omega_r = 0 \tag{7.6}$$

From (7.6) together with (5.2), (7.5) it follows that

$$(M/r)_r = 0, \qquad h_r = 0. \tag{7.7}$$

The compatibility condition of equations (5.2) and (7.5), which was obtained in [1] is

$$D_0 h = r^{-1} M(1 + h^2 -). \tag{7.8}$$

Thus, the system of magnetohydrodynamics equations (3.1) on the solution (4.3) in the case of radial magnetic field is reduced to the invariant subsystem and overdetermined system in involution for the non-invariant function ω. The invariant subsystem consists of equations (5.3) (with $N = 0$), equations (7.1)–(7.4) and also equations (7.7), (7.8). Integration of equations (7.1)–(7.4), (7.7) gives

$$M = a(t)r, \qquad U = -r\frac{a'(t)}{2a(t)}, \qquad H = \frac{1}{r^2 b(t)}, \qquad h = -\frac{b'(t)}{a(t)b(t)}. \tag{7.9}$$

Here $a(t)$ and $b(t)$ are arbitrary functions. The expression for function ρ follows from (5.1)

$$\rho = \frac{a(t)^{3/2}}{b(t)}\rho_0\big(r\sqrt{a(t)}\big). \tag{7.10}$$

Here ρ_0 is an arbitrary function. Equation (7.8) produces the restriction for functions $a(t)$ and $b(t)$:

$$ab'' - a'b' + a^3 b = 0, \tag{7.11}$$

which can be integrated as

$$\left(\frac{b'}{a}\right)^2 + b^2 = C_1^2, \qquad C_1 = \text{const}.$$

The latter is also integrated in the form

$$b(t) = C_1\cos(\tau + C_2), \qquad \tau = \int_0^t a(s)\,\mathrm{d}s, \qquad C_1, C_2 = \text{const}.$$

Using the transformations of time and space dilatation and time shift, admitted by the MHD equations (3.1), one can make $C_1 = 1$, $C_2 = 0$.

Only two equations (5.3) of the invariant subsystem remain. After substitution of representations (7.9), (7.10) the latter equations form an overdetermined system for function $p(t, r)$. For convenience let us introduce new variables (Lagrangian coordinates) $(t, r) \to (t, \lambda)$, $\lambda = r\sqrt{a}$. In the new variables equations (5.3) become

$$\begin{gathered} p_\lambda = \lambda\rho_0(\lambda)\alpha(t), \qquad p_t + A(p, \rho)\beta(t) = 0, \\ \alpha(t) = \frac{\sqrt{a}}{b}\left(\left(\frac{a'}{2a}\right)' - \left(\frac{a'}{2a}\right)^2 + a^2\right), \qquad \beta(t) = \frac{b'}{b} - \frac{3a'}{2a}. \end{gathered} \tag{7.12}$$

The compatibility condition of equations (7.12) is

$$\frac{\rho_0'(\lambda)}{\lambda\rho_0(\lambda)}A_\rho\frac{a(t)^{3/2}}{b(t)}\beta(t) + \alpha(t)\beta(t)A_p + \alpha'(t) = 0. \tag{7.13}$$

The variables t and λ in (7.13) must be separated. In the general case it produces the restrictions on the function $A(p, \rho)$, i.e. on the state equation of the fluid. Let us observe a case of the polytropic state equation $p = S\rho^\gamma$. Here S is an entropy and γ is a polytropic exponent. In this case $A(p, \rho) = \gamma p$. Equation (7.13) becomes

$$\gamma\alpha(t)\beta(t) + \alpha'(t) = 0.$$

Integration of this equation using the expression for $\beta(t)$ (7.12), we obtain

$$\alpha = \frac{\kappa}{4}\left(\frac{a^{3/2}}{b}\right)^\gamma, \qquad \kappa = \text{const}.$$

Substitution of the expression for $\alpha(t)$ from (7.12) into the latter equation gives an equation for the function $a(t)$:

$$2aa'' - 3a'^2 + 4a^4 = \kappa\frac{a^{3(\gamma+1)/2}}{b^{\gamma-1}}. \tag{7.14}$$

An expression for the pressure p follows from the system (7.12):

$$p = p_0(\lambda)a(t)^{3/2\gamma}b(t)^{-\gamma}, \qquad p_0(\lambda) = \int \lambda\rho_0(\lambda)\,\mathrm{d}\lambda.$$

The overdetermined system for the function ω consists of equations (5.2), (7.5), (7.6). This system is in involution of the solutions of the invariant subsystem. Moreover, it can be completely integrated. The general solution $\omega = \omega(t, \theta, \varphi)$ of system (5.2), (7.5), (7.6) is determined in implicit form as

$$F(\eta, \zeta) = 0, \tag{7.15}$$

where invariants η and ζ are

$$\begin{aligned} \eta &= \cos\tau\sin\theta\cos\omega - \sin\tau\cos\theta, \\ \zeta &= \varphi + \arctan\frac{\sin\omega\cos\tau}{\cos\theta\cos\omega\cos\tau + \sin\theta\sin\tau}. \end{aligned} \tag{7.16}$$

Summarizing all, we can formulate the following statement.

Theorem 1. *The solution of magnetohydrodynamics equations (3.1) of special vortex type (4.3) for the polytropic state equation $p = S\rho^\gamma$ and for the pure radial magnetic field is determined by the formulae*

$$\begin{array}{llll} M = a(t)r, & U = -r\dfrac{a'(t)}{2a(t)}, & H = \dfrac{1}{r^2 b(t)}, & h = -\dfrac{b'(t)}{a(t)b(t)}, \\ \rho = \dfrac{a(t)^{3/2}}{b(t)}\rho_0(\lambda), & p = p_0(\lambda)a(t)^{3/2\gamma}b(t)^{-\gamma}, & p_0(\lambda) = \displaystyle\int \lambda\rho_0(\lambda)\,\mathrm{d}\lambda, & \\ \lambda = r\sqrt{a(t)}, & b(t) = \cos\tau, & \tau = \displaystyle\int_0^t a(s)\,\mathrm{d}s. & \end{array} \tag{7.17}$$

Here the function $a(t)$ satisfies equation (7.14). The non-invariant function ω is implicitly determined by formulae (7.15), (7.16) with function τ, defined by formulae (7.17).

8. Magnetic field with zero radial component

Here we observe case 2 from (6.1), namely $H = 0$, $N \neq 0$. From (5.9) after cancellation of the common factor N we have

$$-\sin\theta\sin(\sigma + \omega)\omega_\theta + \cos(\sigma + \omega)\omega_\varphi + \cos\theta\cos(\sigma + \omega) = 0. \tag{8.1}$$

Next, we calculate the compatibility condition of equations (5.2) and (8.1). The compatibility conditions are calculated in the usual way. We sought the solution $\omega = \omega(t, r, \theta, \varphi)$ of equations (5.2) and (8.1) in an implicit form: $\Phi(t, r, \theta, \varphi, \omega) = 0$. Then the equations are represented as vanishing action of linear differential operators Ξ_1 and Ξ_2 on function Φ:

$$\Xi_1\Phi = 0, \qquad \Xi_2\Phi = 0,$$

where

$$\Xi_1 = \sin\theta\sin\omega\partial_\theta - \cos\omega\partial_\varphi + (h\sin\theta + \cos\theta\cos\omega)\partial_\omega,$$
$$\Xi_2 = \sin\theta\sin(\sigma+\omega)\partial_\theta - \cos(\sigma+\omega)\partial_\varphi + \cos\theta\cos(\sigma+\omega)\partial_\omega.$$

Compatibility of equations (5.2), (8.1) means that commutator of operators Ξ_1 and Ξ_2 can be expressed as its linear combination. The calculation of the commutator gives

$$\Xi_3 = [\Xi_1, \Xi_2] = h\sin^2\theta\cos(\sigma+\omega)\partial_\theta + (h\sin\theta\sin(\sigma+\omega) + \cos\theta\sin\sigma)\partial_\varphi$$
$$- (h\sin 2\theta\sin(\sigma+\omega) + \cos 2\theta\sin\sigma)\partial_\omega. \qquad (8.2)$$

Operators Ξ_1, Ξ_2, Ξ_3 are linearly dependent if and only if the following determinant vanishes

$$(h^2 + \sin^2\sigma)\sin^3\theta.$$

From the latter it follows that $h = \sigma = 0$. This implies that we deal with a partial case of the more general situation $\sigma = 0$, which will be investigated separately.

9. Radial fluid motion

Let $M = 0, UN \neq 0$. In this case we do not need the function σ, which differentiates the angles Ω and Σ (see equations (4.1), (4.3)), since the tangential component of velocity vector field vanishes. Therefore we will consider $\sigma(t, r) = 0$ throughout over this section.

Equation (5.4) gives

$$H(rN)_r = 0. \qquad (9.1)$$

Linear combination of equations (5.5), (5.6) produces two relations for function ω:

$$rHN^{-1}\cos\omega\omega_r + \omega_\theta + rH_t(NU)^{-1}\sin\omega = 0, \qquad (9.2)$$
$$rHN^{-1}\sin\theta\sin\omega\omega_r + \omega_\varphi + \cos\theta - rH_t(NU)^{-1}\sin\theta\cos\omega = 0. \qquad (9.3)$$

Equation (5.6), taking into account (5.2), gives

$$rH_t + hUN = 0. \qquad (9.4)$$

From equation (5.7) it follows that

$$D_0N + N(U_r + r^{-1}U) = 0. \qquad (9.5)$$

Equation (5.8) gives another equation for the non-invariant function ω

$$D_0\omega = 0. \qquad (9.6)$$

Finally, from equations (5.9) and (5.2) we have

$$rH_r + 2H - hN = 0. \qquad (9.7)$$

Note that according to equations (9.2), (9.3) the case $H = 0$ leads to contradictory equations for ω. This means that we can cancel the factor H in equation (9.1) and find N as

$$N = a(t)/r, \qquad (9.8)$$

with an arbitrary function $a(t)$. Equation (9.5) gives an expression for U:

$$U = -\frac{a'(t)r + b'(t)}{a(t)}, \tag{9.9}$$

with an arbitrary function $b(t)$. Elimination of function h from (9.4), (9.7) and substitution of expressions for N and U allow one to determine the function H as

$$H = \frac{f(ra(t) + b(t))}{r^2}, \tag{9.10}$$

with an arbitrary function f. The argument $\lambda = a(t)r + b(t)$ of the function f is a Lagrangian invariant, i.e. it conserves along the particle's trajectory: $D_0\lambda = 0$. From (9.6) it follows that $\omega = \omega(\lambda, \theta, \varphi)$. Finally, from (9.4) we obtain

$$h = f'(\lambda). \tag{9.11}$$

Substitution of the obtained representations into equations (9.2), (9.3) gives

$$f(\lambda)\cos\omega\omega_\lambda + \omega_\theta - f'(\lambda)\sin\omega = 0, \tag{9.12}$$

$$f(\lambda)\sin\theta\sin\omega\omega_\lambda + \omega_\varphi + \cos\theta + f'(\lambda)\sin\theta\cos\omega = 0. \tag{9.13}$$

The compatibility condition for these two equations could be found in the same way, as in the previous section. We have

$$ff'' = f'^2 + 1. \tag{9.14}$$

From (9.14) we find that

$$f(\lambda) = C_1\cosh(C_1\lambda + C_2), \qquad C_1, C_2 = \text{const}. \tag{9.15}$$

Note that one can make $C_1 = 1$, $C_2 = 0$ using the time and space dilatation and arbitrariness in the choice of function $b(t)$.

Now it remains to integrate an invariant subsystem, which is determined by equations (5.1), (5.3). The density ρ is found from (5.1) in the form

$$\rho = a(t)\frac{\rho_0(\lambda)}{r^2}, \tag{9.16}$$

with an arbitrary function ρ_0. Two equations (5.3) form an overdetermined system for pressure p:

$$\begin{aligned}&\left(-aa'' + 2a'^2\right)r - ab'' + 2a'b' + \frac{r^2a}{\rho_0(\lambda)}p_r = 0,\\ &D_0p - A(p,\rho)\left(\frac{3a'}{a} + \frac{2b'}{ra}\right) = 0.\end{aligned} \tag{9.17}$$

We observe here only the case of the polytropic state equation $A = \gamma p$. In this case the second equation of (9.17) is integrated as

$$p = p_0(\lambda)a(t)^{3\gamma}(\lambda - b(t))^{2\gamma}. \tag{9.18}$$

Substitution of this representation into the first equation of (9.17) gives a rather cumbersome compatibility condition. It significantly simplifies if $b(t) = 0$. Then, equations (9.17) can be transformed to Lagrangian variables (t, λ) and written in the form

$$p_t = 3\gamma p\frac{a'}{a}, \qquad p_\lambda = \frac{\rho_0(\lambda)}{\lambda}\left(a'' - 2\frac{a'^2}{a}\right) = \frac{\rho_0(\lambda)}{\lambda}\alpha(t). \tag{9.19}$$

From the compatibility conditions of (9.19), we obtain

$$\alpha = \kappa a^{3\gamma} \Rightarrow a'' - 2\frac{a'^2}{a} = \kappa a^{3\gamma}, \qquad \kappa = \text{const}.$$

This equation is autonomous, therefore it can be integrated once:

$$a'^2 = a^4 C + \frac{2\kappa}{3(\gamma-1)} a^{3\gamma+1}, \qquad C = \text{const.} \tag{9.20}$$

Equation (9.20) also can be integrated in implicit form. The pressure is expressed in the form

$$p = \kappa p_0(\lambda) a^{3\gamma}, \qquad p_0(\lambda) = \int \frac{\rho_0(\lambda)}{\lambda}\,\mathrm{d}\lambda.$$

Now we can integrate equations (9.12), (9.13) for the non-invariant function ω. The general solution $\omega = \omega(\lambda, \theta, \varphi)$ is implicitly defined by the formula

$$F(\eta, \zeta) = 0, \tag{9.21}$$

where F is an arbitrary function and

$$\eta = \frac{\sin\theta\cos\omega}{\cosh\lambda} - \cos\theta\tanh\lambda, \qquad \zeta = \varphi + \arctan\frac{\sin\omega}{\cos\theta\cos\omega + \sin\theta\sinh\lambda}. \tag{9.22}$$

Note that expressions (7.16) and (9.22) formally coincide if we put $\tau = \arcsin(\tanh\lambda)$. It is the reflection of the fact that cases 1 and 3 in classification (6.1) are degenerate subcases of a more general situation 4. The latter is investigated in the next section. The result of this section is formulated in the following statement.

Theorem 2. *The solution of magnetohydrodynamics equations (3.1) of special vortex type (4.3) for the pure radial fluid motions is determined by the formulae*

$$\begin{aligned} &U = -\frac{a'(t)r + b'(t)}{a(t)}, \qquad && H = \frac{\cosh\lambda}{r^2}, \qquad && N = \frac{a(t)}{r}, \\ &h = \sinh\lambda, && \rho = \frac{\rho_0(\lambda)a(t)}{r^2}, && \lambda = a(t)r + b(t). \end{aligned} \tag{9.23}$$

Here $\rho_0(\lambda)$ is an arbitrary function. For the polytropic state equation $p = S\rho^\gamma$, the pressure is determined by

$$p = p_0(\lambda)a(t)^{3\gamma}(\lambda - b(t))^{2\gamma}, \qquad p_0(\lambda) = \int \frac{\rho_0(\lambda)}{\lambda}\,\mathrm{d}\lambda. \tag{9.24}$$

Functions $a(t)$ and $b(t)$ can be obtained from equations (9.17) after substitution of (9.24) and separation of variables. In the partial case $b(t) = 0$, function $a(t)$ satisfies equation (9.20).

The non-invariant function $\omega = \omega(\lambda, \theta, \varphi)$ is implicitly determined by formulae (9.21), (9.22).

10. Coplanar magnetic field, velocity and radius vector

Now we observe case 4 of classification (6.1). It corresponds to the case when the radius vector of any particle belongs to the plane defined by its velocity and magnetic field vectors. Let us rewrite equations (5.4)–(5.9) for the case $\sigma = 0$. From (5.4) there follows an invariant equation

$$D_0(rM) - \frac{H(rN)_r}{\rho} = 0. \tag{10.1}$$

Equation (5.6), taking into account (5.2), gives

$$rH_t + hUN - hMH = 0. \tag{10.2}$$

From equation (5.7), we obtain

$$D_0(rN) + rNU_r - (rMH)_r = 0. \tag{10.3}$$

Another equation of invariant subsystem follows from (5.9) taking into consideration (5.2):

$$(r^2H)_r - rhN = 0. \tag{10.4}$$

The overdetermined subsystem for the non-invariant function ω consists of equation (5.2) and also from two equations, which are obtained from (5.5) and (5.8):

$$M\sin\theta\omega_t + (MU - \rho^{-1}HN)\sin\theta\omega_r + Q\sin\theta\cos\omega\omega_\theta + Q\sin\omega\omega_\varphi + Q\cos\theta\sin\omega = 0, \tag{10.5}$$

$$N\omega_t + (NU - HM)\omega_r = 0, \qquad Q = r^{-1}\rho^{-1}(M^2\rho - N^2). \tag{10.6}$$

Multiplication of equations (10.5) and (10.6) by N and $M\sin\theta$ respectively and subtraction give the following

$$Q(rH\sin\theta\omega_r + N\sin\theta\cos\omega\omega_\theta + N\sin\omega\omega_\varphi + N\cos\theta\sin\omega) = 0. \tag{10.7}$$

From (10.7) we have two possible cases: $Q = 0$ and $Q \neq 0$. We will consider these two cases separately.

10.1. The special subcase

Let $Q = 0$. For the convenience we denote $\rho = Z(t, r)^2$. Then we have

$$N = MZ. \tag{10.8}$$

Further we will assume $M \neq 0$. Equations (10.5) and (10.6) coincide and are reduced to

$$D_0\omega - \frac{H}{Z}\omega_r = 0. \tag{10.9}$$

The compatibility condition of (5.2), (10.9) is

$$D_0h - \frac{H}{Z}h_r = 0. \tag{10.10}$$

For further investigation it is convenient to introduce the new notations

$$U_1 = U - HZ^{-1}, \qquad M_1 = r^{-1}M, \qquad H_1 = r^2H, \qquad Z_1 = r^2Z.$$

Equations (10.1)–(10.4), (10.10) can be rewritten as

$$\begin{array}{lll} \bar{D}_0M_1 + M_1U_{1r} = 0, & \bar{D}_0H_1 = 0, & \bar{D}_0Z_1 = 0, \\ H_{1r} - hM_1Z_1 = 0, & \bar{D}_0h = 0, & \bar{D}_0 = \partial_t + U_1\partial_r. \end{array} \tag{10.11}$$

System (10.11) must be extended by equation (5.1), which in the new notation is equivalent to

$$U_{1r} + \frac{r^2H_1}{Z_1^2}\left(\frac{Z_1}{r^2}\right)_r - \frac{2U_1}{r} = 0. \tag{10.12}$$

From (10.11) there follows the functional dependence of functions H_1, Z_1 and h. Let us observe the first case $H_{1r} \neq 0$. In this case one can represent the functional dependence as

$$Z_1 = f(H_1), \qquad h = g(H_1). \tag{10.13}$$

Substitution of relations (10.13) into the fourth equation of (10.11) allows one to express the function M_1 as

$$M_1 = \frac{H_{1r}}{f(H_1)g(H_1)}. \tag{10.14}$$

Substitution of expression (10.14) into the first equation of (10.11) gives

$$-\frac{H_1}{f(H_1)g(H_1)}U_{1r} + \frac{H_{1r}}{f(H_1)g(H_1)}U_{1r} = 0.$$

The latter equation produces the following possibilities

$$(a) \quad U_1 = U_1(t), \qquad (b) \quad H_1 = H_0(t)\mathrm{e}^r.$$

One can check that case (b) leads to $U_1 = U_1(t)$, i.e. it is a subcase of (a). For $U_1 = U_1(t)$ the first equation of (10.11) gives $\bar{D}_0 M_1 = 0$, which means that all functions M_1, H_1, Z_1 and h depend on one function $\mu(t, r)$, which satisfies the equation

$$\mu_t + U_1\mu_r = 0.$$

Thus, the general solution of equations (10.11) under the assumption $H_{1r} \neq 0$ is given by the formulae

$$\begin{aligned} &U_1 = U_1(t), \qquad H_1 = H_1(\mu), \qquad Z_1 = Z_1(\mu), \\ &h = h(\mu), \qquad M_1 = \frac{H_{1\mu}}{h(\mu)Z_1(\mu)}, \end{aligned} \tag{10.15}$$

where U_1 is an arbitrary function of t and H_1, Z_1, h are arbitrary functions of

$$\mu = r - \int U_1(t)\,\mathrm{d}t.$$

Now we should observe equation (10.12). The analysis of equation (10.12) on the solution (10.15) allows us to formulate the following statement.

Theorem 3. *The overdetermined system of equations (10.11), (10.12) with additional assumption $H_{1r} \neq 0$ has only the following two sets of solutions.*

1. $U_1 = U_0\,\mathrm{e}^{C_1 t}$, $H_1 = C_1 C_2 \mu^3$, $Z_1 = C_2\mu^2$, $M_1 = M_1(\mu)$, $h = 3C_1 M_1^{-1}$, $\mu = r - \frac{U_0}{C_1}\,\mathrm{e}^{C_1 t}$.
2. $U_1 \equiv 0$, $Z_1 = Cr^2$, $H_1 = H_1(r)$, $h = h(r)$, $M_1 = H_1'(r)(Ch(r)r^2)^{-1}$.

Here C_1, C_2 are arbitrary non-zero constants and H_1, M_1, h are arbitrary functions of its arguments.

Let us now observe the case $H_{1r} = 0$ in (10.11). The assumption implies that

$$H_1 = H_0 = \text{const}, \qquad h \equiv 0. \tag{10.16}$$

The overdetermined system (10.11), (10.12) is reduced to the following

$$\begin{aligned} &\bar{D}_0 M_1 + M_1 U_{1r} = 0, \qquad \bar{D}_0 Z_1 = 0, \qquad \bar{D}_0 = \partial_t + U_1\partial_r, \\ &U_{1r} + \frac{r^2 H_0}{Z_1^2}\left(\frac{Z_1}{r^2}\right)_r - \frac{2U_1}{r} = 0. \end{aligned} \tag{10.17}$$

Equations (10.17) form a well-defined system for functions U_1, M_1 and Z_1.

Besides, there is an overdetermined system of two equations (5.3) for function p, which must be observed on the obtained solutions of equations (10.11), (10.12). The compatibility of the latter system is not investigated at the moment.

10.2. The general case

Here $Q \neq 0$. The overdetermined system for function ω in addition to equation (5.2) includes two equations, which follow from (10.5), (10.6):

$$N\omega_t + (NU - HM)\omega_r = 0, \tag{10.18}$$

$$rH\cos\omega\omega_r + N\omega_\theta - hN\sin\omega = 0. \tag{10.19}$$

The compatibility condition of (5.2) and (10.18) is

$$ND_0h - HMh_r = 0. \qquad (10.20)$$

Another compatibility condition for equations (5.2), (10.19) is

$$rHh_r = N(1 + h^2). \qquad (10.21)$$

Equations (10.18), (10.19) for $N \neq 0$ are compatible by virtue of invariant equations. The case $N = 0$ was investigated earlier.

Thus, the invariant system is composed of the following equations: (5.1), (5.3), (10.1)–(10.4), (10.21), (10.20). It is an overdetermined system of nine equations for seven sought functions M, H, ρ, N, U, h, p. This system must be completed to involution. Further, we will assume $N \neq 0$ since the opposite possibility was completely investigated earlier.

Note that equations (10.20), (10.21) allow us to express both first-order derivatives of $h(t, r)$ in terms of invariant functions such that the right-hand sides of expressions do not involve any other derivatives. The equations (10.2), (10.4) allow us to do the same for function H. One can check that calculation of mixed second-order derivatives of each function by two methods gives the same result by virtue of the equations of the invariant system. This means that the invariant system is already in involution.

It is convenient to use the following notations:

$$M_1 = r^{-1}M, \qquad H_1 = r^2H, \qquad N_1 = rN, \qquad h = \tan\tau. \qquad (10.22)$$

Equations (10.2), (10.4), (10.20) and (10.21) can be rewritten as

$$\begin{aligned} &D_0\tau = M_1, && H_1\tau_r = N_1, \\ &D_0H_1 = M_1H_1\tan\tau, && H_{1r} = N_1\tan\tau. \end{aligned} \qquad (10.23)$$

The last two equations of (10.23) are equivalent to the integral $H_1\cos\tau = C$ with an arbitrary constant C. One can make $C = 1$ using the dilatation transformation, which is admitted by the initial system (3.1), i.e. without loss of generality we can assume

$$H_1 = (\cos\tau)^{-1}. \qquad (10.24)$$

The invariant system becomes the following:

$$\begin{aligned} &D_0M_1 + \frac{2}{r}UM_1 - \frac{1}{r^4\rho\cos\tau}N_{1r} = 0, && D_0 = \partial_t + U\partial_r, \\ &D_0N_1 + N_1U_r - \frac{1}{\cos\tau}M_{1r} - M_1N_1\tan\tau = 0, && \\ &D_0p + A(p,\rho)\left(U_r + \frac{2}{r}U - M_1\tan\tau\right) = 0, && \\ &D_0U + \frac{1}{\rho}p_r + \frac{N_1N_{1r}}{r^2\rho} - rM_1^2 = 0, && \tau_r = N_1\cos\tau, \\ &D_0\rho + \rho\left(U_r + \frac{2}{r}U - M_1\tan\tau\right) = 0, && D_0\tau = M_1. \end{aligned} \qquad (10.25)$$

The overdetermined system (10.25) of seven equations for six sought functions is in involution. It is also worth noting that function τ is determined from system (10.25) with arbitrariness in one constant. In fact, the initial data of the well-posed Cauchy problem for system (10.25) are

$$\begin{aligned} &M_1(0, r) = m(r), && N_1(0, r) = n(r), && p(0, r) = p_0(r), \\ &U(0, r) = u_0(r), && \rho(0, r) = \rho_0(r), && \tau(0, r_0) = \tau_0. \end{aligned} \qquad (10.26)$$

Here m, n, p_0, u_0, ρ_0 are arbitrary functions of r; τ_0 is a constant. Therefore function $\tau(t, r)$ is defined by its value on a fixed sphere $r = r_0$ at initial time $t = 0$.

We now turn to finding the non-invariant function ω. Equations for ω are (5.2), (10.18) and (10.19). In the new notations (10.22), we can rewrite it as

$$\begin{aligned}
&N_1\omega_t + (N_1 U - \cos^{-1}\tau M_1)\omega_r = 0,\\
&\cos\omega\omega_r + N_1\cos\tau\omega_\theta - N_1\sin\tau\sin\omega = 0,\\
&\sin\theta\sin\omega\omega_\theta - \cos\omega\omega_\varphi - \tan\tau\sin\theta - \cos\theta\cos\omega = 0.
\end{aligned}\tag{10.27}$$

The latter system is also in involution on the solutions of the invariant subsystem (10.25). Equations (10.27) can be integrated; its general solution is implicitly determined by the same formula as in the previous cases

$$F(\eta, \zeta) = 0,\tag{10.28}$$

where

$$\begin{aligned}
&\eta = \sin\theta\cos\omega\cos\tau - \cos\theta\sin\tau,\\
&\zeta = \varphi + \arctan\frac{\sin\omega\cos\tau}{\cos\theta\cos\omega\cos\tau + \sin\theta\sin\tau}.
\end{aligned}\tag{10.29}$$

The result of this section can be formulated as follows.

Theorem 4. *The solution of magnetohydrodynamics equations (3.1) of special vortex type (4.3) for the case of coplanar radius vector, velocity and magnetic field vectors is determined by the invariant equations (10.25) and by the implicit formulae (10.28), (10.29) for the non-invariant function* $\omega = \omega(t, r, \theta, \varphi)$.

11. Discussions

The mathematical analysis of the singular vortex in ideal magnetohydrodynamics reveals the following facts.

- There exists a non-trivial $O(3)$ partially invariant solution of the ideal MHD equations.
- An irreducible solution exists only in the case when the velocity, magnetic field and radius vectors of any particle are coplanar.
- The maximal arbitrariness of the solutions is six arbitrary functions of one argument.
- The description of the fluid flow, governed by the obtained solution, is divided into two parts. The first one is a reduced system (10.25) of PDEs with two independent variables, which describes the dependence of all functions on time and on radial coordinate. The second part of the solution is an implicit equation (10.28) for function ω, which determines the vector field on the spheres $r = \text{const}$. Combination of these two parts of the solution gives a description of the plasma motion.

The further investigation is separated into two branches. The first one is the analysis of the invariant system (10.25). The latter, being itself a system of partially differential equations, serves as a subject of symmetry analysis. The invariant solutions of system (10.25) are described by ordinary differential equations and could be observed in detail. The second branch of investigation is a description of the vector field on the sphere $r = \text{const}$, defined by the implicit equation (10.28). This problem is non-trivial; its solution leads to the analysis of the wave fronts and the curve's caustic on the sphere. Besides, the physical description of the plasma flow governed by the singular vortex solution should be given. The designated investigations will be presented soon in a separate article.

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